So far in our discussion of rotations in 3D we have encountered scalars, vectors and tensors. These are 0, 3 and higher dimensional representations of the rotation group.

What about a 2D representation of 3D rotations?

We would need 2×2 matrices satisfying $[g_i, g_j] = i\epsilon^{ijk} g_k$ where $\underbrace{i,j,k=1,2,3}_{3D}$ $\quad [g_1,g_2]=ig_3$, $[g_2,g_3]=ig_1$, $[g_3,g_1]=ig_2$

These work: $g_{R_{YZ}} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$ $\quad g_{R_{ZX}} = \begin{pmatrix} 0 & -i/2 \\ i/2 & 0 \end{pmatrix}$ $\quad g_{R_{XY}} = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$

$\quad = \frac{1}{2}\sigma_x \qquad = \frac{1}{2}\sigma_y \qquad = \frac{1}{2}\sigma_z$ where $\sigma_x, \sigma_y, \sigma_z$ are the Pauli spin matrices

Now we can build: $R_{YZ}(\theta) = e^{i g_{R_{YZ}}\theta} = \begin{pmatrix} \cos(\frac{\theta}{2}) & i\sin(\frac{\theta}{2}) \\ i\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix}$ and similarly for R_{ZX} and R_{XY}.

satisfy $U^+U = \mathbb{I}$ and $\det U = +1$ $\Big\}$ $SU(2)$ which act on complex 2-component spinors χ.

Often we write $\chi \to \chi' = e^{\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}}\chi$ $\qquad$ Note: We will not use spin indices in this class, so we will rely on matrix manipulations.

So $SO(3) \sim SU(2)$, at least near the identity (which is all the Lie algebra knows about).

Globally however there is a difference:

$SO(3)\quad R_x(2\pi) = \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix} = \mathbb{I}$

$SU(2)\quad R_x(2\pi) = \begin{pmatrix} -1&0\\0&-1 \end{pmatrix} = -\mathbb{I}$

$SU(2)$ is called the double-cover of $SO(3)$

of course $R_x(4\pi) = \mathbb{I}$ for both!

There is a certain sense in which spinors and $SU(2)$ probes geometry more deeply than coordinates, scalars, vectors, $SO(3)$, etc.
By "probe more deeply" I mean they contain more information. Sometimes people say that spinors know about the square root of the geometry.
In fact if we consider the anti-commutator of the Pauli matrices we find: $\underbrace{\{\sigma_i, \sigma_j\} = 2\delta_{ij} I_{2\times 2}}_{\text{A Clifford algebra.}}$

Example: $\sigma_x\sigma_y + \sigma_y\sigma_x = \begin{pmatrix} 0&1\\1&0 \end{pmatrix}\begin{pmatrix} 0&-i\\i&0 \end{pmatrix} + \begin{pmatrix} 0&-i\\i&0 \end{pmatrix}\begin{pmatrix} 0&1\\1&0 \end{pmatrix} = \begin{pmatrix} i&0\\0&-i \end{pmatrix} + \begin{pmatrix} -i&0\\0&i \end{pmatrix} = \begin{pmatrix} 0&0\\0&0 \end{pmatrix}$ as expected since $\delta_{xy} = 0$

$\sigma_y\sigma_y + \sigma_y\sigma_y = 2\sigma_y\sigma_y = 2\begin{pmatrix} 0&-i\\i&0 \end{pmatrix}\begin{pmatrix} 0&-i\\i&0 \end{pmatrix} = 2\begin{pmatrix} 1&0\\0&1 \end{pmatrix}$ as expected since $\delta_{yy} = 1$

It might seem silly, but recall that $\delta_{ij} = \begin{pmatrix} 1&&\\&1&\\&&1 \end{pmatrix}$ is the metric of $\mathbb{R}^3$. This will come in handy later.

Another illustration of this is a lesson from QM: If we only have integer spin states at our disposal, $\{0,1,2,\cdots\}$ then by combining spins we can only ever build more integer spin states. However if we allow ½ integer spin states, then we can build ½ or whole integer states just using ½ spin states, e.g. $\frac{1}{2} - \frac{1}{2} = 0$, $\frac{1}{2} + \frac{1}{2} = 1$.

To finish up, we need to determine how to build an invariant (for Lagrangians) out of spinors

Following our usual recipe: If $\chi \to \chi' = e^{\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}}\chi$ and $\tilde{\chi} \to \tilde{\chi}' = \left(e^{\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}}\right)^{-1^{+}}\tilde{\chi}$

then $\tilde{\chi}^{+}\chi$ is invariant. ($+ \leftarrow *T$)

But recall how we form $\tilde{\chi}$ from χ: $\tilde{\chi} = (g\chi)$ where $\left(e^{\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}}\right)^{+} g\, e^{\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}} = g$

However for SU(2) we already know that $U^{+}U = 1$ so $g = I$ and

we can say $\tilde{\chi} = (g\chi) = \chi$ and then $\chi^{+}\chi$ is invariant!

Note: All of the σ matrices are Hermitian, i.e. $\sigma_i^{+} = \sigma_i$, $\vec{\theta}$ is real so

$$U^{+} = \underbrace{\left(e^{\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}}\right)^{+}}_{\text{You can see more explicitly by Taylor expanding}} = e^{-\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}} = U^{-1}$$ This will <u>not</u> be the case later!

$$\left[I + \left(\tfrac{i}{2}\vec{\sigma}\cdot\vec{\theta}\right) + \tfrac{1}{2}\left(\tfrac{i}{2}\vec{\sigma}\cdot\vec{\theta}\right)\left(\tfrac{i}{2}\vec{\sigma}\cdot\vec{\theta}\right) + \cdots\right]^{+}$$

$$= I + \left(-\tfrac{i}{2}\vec{\sigma}^{+}\cdot\vec{\theta}\right) + \tfrac{1}{2}\left(-\tfrac{i}{2}\vec{\sigma}^{+}\cdot\vec{\theta}\right)\left(-\tfrac{i}{2}\vec{\sigma}^{+}\cdot\vec{\theta}\right) + \cdots$$
$$= I + \left(-\tfrac{i}{2}\vec{\sigma}\cdot\vec{\theta}\right) + \tfrac{1}{2}\left(-\tfrac{i}{2}\vec{\sigma}\cdot\vec{\theta}\right)\left(-\tfrac{i}{2}\vec{\sigma}\cdot\vec{\theta}\right) = e^{-\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}}$$

Now it is time to repeat this procedure for special relativity.

The Lorentz transformations as they act on coordinates/vectors form $SO(1,3)$ so let's explore its algebra.

We expect 6 generators corresponding to: $\underbrace{R_{yz}, R_{zx}, R_{xy}}, \underbrace{B_{xt}, B_{yt}, B_{zt}}$.

We will call the corresponding generators: $J_1, J_2, J_3 \quad K_1, K_2, K_3$

Fortunately we already know a lot about the J's:

$$J_1 = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&-i \\ 0&0&i&0 \end{pmatrix} \quad J_2 = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&i \\ 0&0&0&0 \\ 0&-i&0&0 \end{pmatrix} \quad J_3 = \begin{pmatrix} 0&0&0&0 \\ 0&0&-i&0 \\ 0&i&0&0 \\ 0&0&0&0 \end{pmatrix} \Rightarrow \underbrace{[J_i, J_j] = i\epsilon^{ijk} J_k}_{\text{From which we can also get } SU(2)}$$

If we take the various boosts and again consider their Taylor expansion, then using the exponential map $B = \exp(iK\delta B)$ we find:

$$K_1 = \begin{pmatrix} 0&-i&0&0 \\ -i&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} \quad K_2 = \begin{pmatrix} 0&0&-i&0 \\ 0&0&0&0 \\ -i&0&0&0 \\ 0&0&0&0 \end{pmatrix} \quad K_3 = \begin{pmatrix} 0&0&0&-i \\ 0&0&0&0 \\ 0&0&0&0 \\ -i&0&0&0 \end{pmatrix}$$

Now is where it gets interesting. By brute force one can show:

$[K_i, K_j] = -i\epsilon^{ijk} J_k$ 2 boosts → rotation

$[J_i, K_j] = i\epsilon^{ijk} K_k$ rotation + boost = boost

Question: Can the boosts alone form a subgroup of $SO(1,3)$? No
What about rotations? Yup

So unfortunately the boosts and rotations of $SO(1,3)$ do not cleanly split from each other.

But...

Let's play an old math/physics trick:

Define $J_{+i} = \frac{1}{2}(J_i + iK_i)$

$J_{-i} = \frac{1}{2}(J_i - iK_i)$

$$\Rightarrow \begin{array}{ll} [J_{+i}, J_{+j}] = i\epsilon^{ijk} J_{+k} & \Rightarrow SO(3) \\ [J_{-i}, J_{-j}] = i\epsilon^{ijk} J_{-k} & \Rightarrow SO(3) \\ [J_{+i}, J_{-j}] = 0 & \Rightarrow \text{These } SO(3) \text{ don't mix.} \end{array}$$

(Then:)

So we find that at least near the identity $SO(1,3) \sim \underbrace{SO(3) \times SO(3)}$

Remember this is <u>not</u> a split into 3 boosts and 3 rotations!!

Now everything so far has been in terms of coordinates (scalars, vectors, tensors, etc.), but we can immediately see how to introduce spinors.

We utilize $SO(1,3) \sim SO(3) \times SO(3) \sim SU(2) \times SU(2)$

Each of these will act on a complex 2 component object, so our total spinor in 4D has 4 complex components!

This is most unfortunate since now we have 4 component vectors and 4 component spinors, but the components mean totally different things. This is only a misfortune in 4D.

	3D	4D	5D	6D	7D	8D	9D	10D
vector	3	4	5	6	7	8	9	10
spinor	2	4	4	8	8	16	16	32

The counting goes: For each <u>independent</u> plane you can define an independent $SU(2)$ w/ a 2-component spinor giving $2^{d/2}$ or $2^{(d-1)/2}$ states depending on d even or odd.

We now need to determine how these 4-component spinors transform and then how to build an invariant.

You might think we could just use the 4x4 matrices we already have for K_i and J_i, but remember these act on coordinate related quantities not spinors.

So what should we use? There are numerous ways to get at the answer, but we will use the deepest based on the idea of the square root of the geometry.

Recall: $\underbrace{\{\sigma_i, \sigma_j\} = 2\delta_{ij}}_{\text{metric in 3D}} I_{2\times2}$ $\Rightarrow \chi \to \chi' = e^{\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}}\chi$

Then perhaps: $\underbrace{\{\gamma^\mu, \gamma^\nu\}}_{\gamma^0,\gamma^1,\gamma^2,\gamma^3} = 2\eta^{\mu\nu} I_{4\times4}$ $\overset{?}{\Rightarrow} \psi \to \psi' = e^{\frac{i}{2}\vec{\gamma}\cdot\vec{\theta}}\psi$

Unfortunately this won't work! We expect 4 γ's, so this is only 4 distinct transformations, but we know there should be 6!
$\uparrow$
$SO(1,3)$

Fortunately the answer is hiding in our bad notation.

If instead of $(\sigma_1, \sigma_2, \sigma_3)$ we think of $(\underbrace{-i[\sigma_2,\sigma_3]}, -i[\sigma_3,\sigma_1], -i[\sigma_1,\sigma_2])$

Rotation around x is really in the y-z plane.

Then we can think of: $\left(-\frac{i}{4}[\gamma^0,\gamma^1], -\frac{i}{4}[\gamma^0,\gamma^2], -\frac{i}{4}[\gamma^0,\gamma^3], -\frac{i}{4}[\gamma^1,\gamma^2], -\frac{i}{4}[\gamma^2,\gamma^3], -\frac{i}{4}[\gamma^3,\gamma^1]\right)$

If we call these $\sigma^{\mu\nu} = \{\sigma^{01}, \sigma^{02}, \sigma^{03}, \sigma^{12}, \sigma^{23}, \sigma^{31}\}$

$\sigma^{01} = -\sigma^{10}$, $\sigma^{02} = -\sigma^{20}$, $\sigma^{03} = -\sigma^{30}$, $\sigma^{12} = -\sigma^{21}$, $\sigma^{23} = -\sigma^{32}$, $\sigma^{31} = -\sigma^{13}$

Then parameterizing the transformation with angles $\{\alpha, \beta, \gamma, \theta, \phi, \psi\} \equiv \{\omega_{01}, \omega_{02}, \omega_{03}, \omega_{12}, \omega_{23}, \omega_{31}\}$

$\omega_{01} = -\omega_{10}$, $\omega_{02} = -\omega_{20}$, $\omega_{03} = -\omega_{30}$, $\omega_{12} = -\omega_{21}$, $\omega_{23} = -\omega_{32}$, $\omega_{31} = -\omega_{13}$

We can write our transformation: $\boxed{\psi \to \psi' = e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\psi}$

Example: Rotation in y-z by ϕ uses $\omega_{\mu\nu} = \{0, 0, 0, 0, \omega_{23} = \phi, 0\}$ giving $\psi \to \psi' = e^{\frac{i}{4}(\sigma^{23}\omega_{23} + \sigma^{32}\omega_{32})}\psi$

or

$\psi' = e^{\frac{i}{2}\sigma^{23}\phi}\psi$

Without any further ado, I present (at least one set of) the Dirac γ matrices:

$$\gamma^0 = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^i = -i\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \qquad \text{w/} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Example: $\gamma^2 = -i\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$

These have some nice properties:

- Recall $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} I_{4\times4}$
- Then $(\gamma^0)^2 = -1$, $(\gamma^i)^2 = 1$
- And $\underbrace{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0}_{\text{or } \gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu}$ if $\mu \neq \nu$ since $\eta^{\mu\nu}$ is diagonal

We can now explicitly form the generators:

$$\sigma^{0i} = -\frac{i}{4}[\gamma^0, \gamma^i] = -\frac{i}{4}\left[\gamma^0\gamma^i - \gamma^i\gamma^0\right]$$

$$= -\frac{i}{4}\left[-i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(-i)\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} + i\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}(-i)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$$

$$= \frac{i}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}$$

Note: We now see why we needed the $\frac{i}{4}$ in the definition. The transformation now reduces to the usual $SU(2)$ transformation on each pair of spinor indices, i.e. $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \begin{matrix} \}\, SU(2) \\ \}\, SU(2) \end{matrix}$

$$\sigma^{ij} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \qquad \text{e.g. } \sigma^{12} = \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$$